The Problem of Routing on Graphs
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1 Main problems

Let $G = (V, E)$ be a connected simple graph, where $V$ and $E$ represent the vertex and edge sets, respectively, of $G$. The number of vertices in $V$ is denoted by $n = |V|$. Initially, each vertex $v$ of $G$ is occupied by a “pebble” (The pebble represents a packet of information) which has a unique destination $\pi(v)$ in $G$ (so that $\pi$ is a permutation of the vertices of $G$). Our task is to simultaneously move a pebble from each vertex $v$ to its destination $\pi(v)$. It can be finished by performing a sequence of moves of the following type: A vertex disjoint set of edges is selected, and the pebbles at each edge’s endpoints are interchanged. We refer this model as a matching routing model, and define the $rt(G)$ to be the minimum number of steps required for any possible permutation $\pi$, so we define the routing number of $G$ by $rt(G) = \max_{\pi} rt(G, \pi)$. Here is one example to illustrate the routing problem: The graph $G$ is given by vertex set $V = \{1, 2, 3, 4\}$, edge set $E = \{(1, 2), (1, 3), (1, 4), (2, 3)\}$. We want to route a permutation $\pi$ which is given by $\pi := \{\pi(1) = 3, \pi(2) = 1, \pi(3) = 4, \pi(4) = 2\}$ (i.e. each pebble $p_i$ needs to be moved to vertex $i$, for $i = 1, 2, 3, 4$). It turns out that we need three steps: 1) swap $p_3$ and $p_4$; 2) swap $p_2$ and $p_4$; 3) swap $p_1$ and $p_2$, thus $rt(G, \pi) \leq 3.$ (The highlight on each graph indicates the step to be performed in the next step)

2 Past results

For a path, Alon, Chung and Graham [1] show that any permutation on a path of $n$-vertices can be routed by at most $n$ steps using the odd-even algorithm.

Among all the connected graphs, trees have the highest routing numbers, since extra edges only provide more flexibility for routing. So it is natural to seek the maximum of $rt(T)$ over $n$-vertex trees. It was shown in [1] by Alon, Chung and Graham that $rt(T) \leq 3n$ for any tree $T$ of $n$ vertices; later $rt(T) \leq 13n/5$ in [4] by Roberts, Symvonis, and Zhang; then $rt(T) \leq 2n$ in [5] [3] [6] by Goddard, Hoyer and Larson, and Zhang; in [6] Zhang also improved the result to $rt(T) \leq 3n/2 + O(\log_3 n)$, which is asymptotically tight by the star trees.

Hypercube networks and Cartesian product networks are very important network systems. It was conjectured in [1] that the routing number of an $n$-dimensional hypercube is $rt(Q_n) = n + 1$, however, the latest result is that $n + 1 \leq rt(Q_n) \leq 2n - 2$ in [7] by Li, Lu and Yang. For the Cartesian product of two graphs, we have only that $rt(G_1 \square G_2) \leq 2rt(G_1) + rt(G_2)$.

In the year 2010, the authors in [7] introduced a variation of routing problem, called fractional routing problem in which each pebble is allowed to be split into multiple pieces during the process of routing. This innovative technique has greatly improved algorithms by reducing the routing number of various types of graphs. In their study, the routing number of a star tree $S_n$ on $n$ vertices is improved from $rt(S_n) = \lfloor 3n/2 \rfloor$ to $rt'(S_n) = n$; for complete bipartite graph $K_{n,n}$, the routing number is improved from $rt(K_{n,n}) = 4$ to $rt'(K_{n,n}) = 3$.

It is certain that using fractional routing method can improve algorithms of routing problems over many other graphs. However, we are not aware of any further research under this new theme. This problem aims to prove bounds for the fractional routing number of certain classes of graphs, improving the bounds of the classical routing number.
2.1 The fractional routing method

The fractional routing method assumes all pebbles have mass 1 and can be split into a same number of smaller pieces during the routing process. After pebble \( p_i \) reaching its destination, all pieces of the same type can be assembled into the pebble \( p_i \). The pieces can be exchanged through a fractional matching at one step. A fractional matching is a mapping

\[
    f : E(G) \rightarrow [0, 1]
\]

satisfying for any vertex \( v \), \( \sum_{N(v)} f(uv) \leq 1 \), here \( N(v) \) is the set of neighbor vertices. For each edge \( uv \), pieces of total mass \( f(uv) \) at \( u \) can be exchanged with pieces of the same total mass \( f(uv) \) at \( v \). Given a permutation \( \pi \), the pebble on \( v_i \) will be labeled as \( p_j \) if \( \pi(i) = j \). We denote the minimum number of steps to route each pebbles \( p_i \) to \( v_i \) as \( rt'(G, \pi) \), thus we aim to find \( rt'(G) = \max\pi rt'(G, \pi) \). Since every matching is a fractional matching, we have

\[
    rt'(G) \leq rt(G).
\]

The following is an example to apply fractional routing method to route a permutation \( \pi \) on a star tree, where \( \pi = \{ \pi(1) = 2, \pi(2) = 1, \pi(3) = 4, \pi(4) = 3, \pi(c) = c \} \). Totally it needs 5 steps, while at least 6 steps if using non-fractional algorithms.

The fractional method can reduce the number of steps for routing problems on star tree. In [7], the authors showed that for any star tree \( S_n \) with \( n \) nodes, \( rt'(S_n) = n \). Note that stars are most difficult graphs to route. This suggests that we can improve the existing results using fractional routing method.

2.2 Open questions

1. For \( n \)-node tree \( T \), prove that \( rt'(T) \leq n(1 + o(1)) \), using fractional routing method;

2. For hypercube \( Q_n \), prove that \( rt'(Q_n) \leq \alpha n \) for any \( \alpha < 2 \);

3. For Cartesian product of two graphs, prove that \( rt'(G_1 \square G_2) \leq rt'(G_1) + rt'(G_2) + O(1) \).

References


